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On the role of rotation in the generation of magnetic fields by fluid motions

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It is generally accepted that the magnetic fields of planets and stars are produced by the self-exciting dynamo process (first proposed by Larmor) and that observed near-alignments of magnetic dipole axes with rotation axes are due to the influence of Coriolis forces on underlying fluid motions. The detailed role of rotation in the generation of cosmical magnetic fields has yet to be elucidated but useful insight can be obtained from general considerations of the governing magnetohydrodynamic equations. A magnetic field $\boldsymbol{B}$ cannot be maintained or amplified by fluid motion $\boldsymbol{u}$ against the effects of ohmic decay unless (*a*) the magnetic Reynolds number $R \equiv UL\bar{\mu}\bar{\sigma}$ is sufficiently large, and (*b*) the patterns of $\boldsymbol{B}$ and $\boldsymbol{u}$ are sufficiently complicated (where U is a characteristic flow speed, L a characteristic length and $\bar{\mu}$ and $\bar{\sigma}$ are typical values of the magnetic permeability and electrical conductivity respectively). Axisymmetric magnetic fields will always decay (a result that suggests that palaeomagnetic and archaeomagnetic data might show evidence that departures from axial symmetry in the geomagnetic field are systematically less during the decay phase of a polarity 'reversal' or 'excursion' than during the recovery phase). Dynamo action is stimulated by Coriolis forces, which promote departures from axial symmetry in the pattern of $\boldsymbol{u}$ when $\boldsymbol{B}$ is weak, and is opposed by Lorentz forces, which increase in influence as $\boldsymbol{B}$ grows in strength. If equilibrium is attained when Coriolis and Lorentz forces are roughly equal in magnitude then the system becomes 'magnetostrophic' and the strengths of the internal and external parts of the field, B_i and B_e respectively, satisfy $B_\mathrm{i} \lesssim B_\mathrm{s} R^{\frac{1}{2}}$ and $B_\mathrm{e} \lesssim B_\mathrm{s} R^{-\frac{1}{2}}$ if $B_\mathrm{s} \equiv (\bar{\rho}(\Omega + UL^{-1})/\bar{\sigma})^{\frac{1}{2}} \approx (\bar{\rho}\Omega/\bar{\sigma})^{\frac{1}{2}}$, ($\bar{\rho}$ being the mean density of the fluid and Ω the angular speed of rotation). The slow and dispersive 'magnetohydrodynamic inertial wave' with a frequency that depends on the square of the Alfvén speed $|\boldsymbol{B}|/(\mu\bar{\rho})^{\frac{1}{2}}$ and *inversely* on Ω exemplifies magnetostrophic flow. Such waves probably occur in the electrically conducting fluid interiors of planets and stars, where they play an important role in the generation of magnetic fields as well as in other processes, such as the topographic coupling between the Earth's liquid core and solid mantle.

1. INTRODUCTION

The magnetic fields of the Earth and Sun and of other magnetic planets and stars are thought to be due to electric currents flowing within their interiors. It is now accepted that these currents are largely maintained against the effects of ohmic dissipation by electromotive forces due to motional induction, as Larmor first pointed out in his pioneering paper on self-exciting fluid dynamos. The fluid motions involved are produced in most (if not all) cases by the action of gravity on density inhomogeneities.

Theoretical studies of the flow of electrically conducting fluids – 'magnetohydrodynamics' or 'hydromagnetics' – are based on the highly nonlinear equations of hydrodynamics, thermodynamics and electrodynamics (see § 3). Dynamo models treated on the basis of all these equations are often referred to us 'magnetohydrodynamic dynamos'. But most progress to date has been made with the study of 'kinematic dynamos' for which the field of fluid flow is postulated *a priori* and non-decaying solutions sought of the electrodynamic equations alone (see § 2).

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The mathematical analysis of dynamo models is complicated by the finding that suitable departures from axial symmetry are required for dynamo action to occur (see (2.10)). This follows from existence theorems in kinematic dynamo theory (for reference see, for example, Moffatt 1978; Parker 1979) and the recent extension of Cowling's theorem (Hide & Palmer 1982) showing quite generally:

Fluid motions cannot prevent the ohmic decay of a magnetic field that retains an axis of symmetry. (1.1)

This result (which suggests, incidentally, that palaeomagnetic and archaeomagnetic data might show evidence that departures from axial symmetry are systematically less during the decay phase of a geomagnetic polarity 'reversal' or 'excursion' than during the growth or recovery phase (see Hide 1981 *a*)) provides a useful starting point for discussing how rotation affects the generation of magnetic fields by the dynamo process.

Order of magnitude estimates of the various terms in the dynamical equations (see § 3 below) show that flows associated with typical natural dynamos are strongly influenced by Coriolis forces due to general rotation. It is through the analysis of this influence of Coriolis forces that the explanation of the near-alignment of the rotation and magnetic dipole axes of the Earth, Jupiter, Saturn, etc., and other properties of the magnetic fields must be sought. But further work on the magnetohydrodynamics of rapidly rotating fluids will be needed before the role of rotation in the production of cosmical magnetic fields can be fully elucidated. Details of investigations in this important but highly mathematical branch of geophysical fluid dynamics together with references to early work can be found in several recent publications (see, for example, Moffatt 1978; Roberts & Soward 1978; Busse 1978, 1979; Parker 1979; Stevenson 1979; Braginskiy 1980; Krause & Rädler 1980; Soward 1982). The purpose of the present paper is to outline certain general properties of flows that are strongly influenced by Coriolis forces due to general rotation and Lorentz forces due to the presence of electric currents within the fluid. These properties follow more or less directly from the governing equations (see §§ 3 and 4) and they can serve among other things as a guide to the more technical literature.

Possibly the most significant of these properties for dynamo studies is the result (see (3.10) below):

Rapid rotation promotes departures from axial symmetry in the pattern of fluid motions when the magnetic field is weak. (1.2)

Coriolis forces can thus stimulate the amplification of a weak magnetic field by producing departures from axial symmetry in the pattern of fluid motions. As the magnetic field increases in strength so does the Lorentz force (see equation (3.4)), and it is possible that the amplification of the magnetic field cannot continue beyond the point at which the Lorentz force is typically comparable in magnitude with the Coriolis force. This hypothesis provides a basis for estimating the ultimate strength obtained by the magnetic field and leads to predictions that are concordant with observations (Hide 1974; see also § 4).

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2. Basic equations: electrodynamics

Consider a connected body of electrically conducting fluid V_0 bounded by a surface S_0 with surface element d$\boldsymbol{S}$. The linkage with S_0 of a magnetic field $\boldsymbol{B}$ that pervades the conducting fluid and the surrounding space is defined as the essentially non-negative quantity

$$N(S_0;\,t) \equiv \iint_{S_0} |\boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S}|. \tag{2.1}$$

In the absence of permanent magnets, $\boldsymbol{B}$ is due entirely to electric currents of density $\boldsymbol{j}$ and in the self-exciting dynamo the electromotive forces that produce these electric currents are provided by motional induction, involving fluid motions within V_0 with Eulerian velocity $\boldsymbol{u}$. By this means some weak adventitious seed field can be amplified and maintained against the effects of ohmic decay. If the fluid motions were suddenly to cease, $N(S_0;\,t)$ would decay on a timescale $O(\tau_d)$ where τ_d is the ohmic decay time based on a characteristic length L of the order of the dimensions of V_0 (see equation (2.11) below). For the Earth's liquid electrically conducting core τ_d lies somewhere between 10^4 and 10^5 years; for the cores of Jupiter and Saturn τ_d could be somewhat longer, possibly 10^6 or 10^7 years, but still short compared with the presumed ages of the magnetic fields of these planets, in excess of 10^9 years.

Dynamo action can be said to occur in a theoretical model when the magnitude and configuration of $\boldsymbol{u}$ and $\boldsymbol{B}$ are such that over the long but otherwise arbitrary interval $t = t_1$ to $t = t_2$ (where $t_2 - t_1$ greatly exceeds the ohmic decay time τ_d (see equation (2.11)),

$$(t_2-t_1)^{-1}\int_{t_1}^{t_2} \{\mathrm{d}N(S_0;\,t)/\mathrm{d}t\}\,\mathrm{d}t \nless 0. \tag{2.2}$$

This criterion has advantages over proposals based on total magnetic energy or equivalent magnetic moment, which can be ambiguous when $\boldsymbol{B}$ has toroidal as well as poloidal components or when the conducting fluid is not incompressible (see Hide 1981*b*; Hide & Palmer 1982). We consider first the electrodynamic equations required in the formulation of theoretical models. These are Gauss's law

$$\nabla\cdot\boldsymbol{B} = 0 \tag{2.3}$$

(which implies, of course, that

$$\iint_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = 0$$

taken over any closed surface S; cf. equation (2.1)), Faraday's law

$$\partial\boldsymbol{B}/\partial t + \nabla\times\boldsymbol{E} = 0, \tag{2.4}$$

and Ampère's law

$$\nabla\times(\mu^{-1}\boldsymbol{B}) = \boldsymbol{j}, \tag{2.5}$$

together with Ohm's law

$$\boldsymbol{j} = \sigma(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}), \tag{2.6}$$

where $\boldsymbol{E}$ is the electric field in the basic frame of reference and $\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}$ is the electric field experienced by a fluid element moving with velocity $\boldsymbol{u}$ relative to that frame. The magnetic permeability μ and electrical conductivity σ are scalars but they may be functions of position and time.

When $\boldsymbol{E}$ and $\boldsymbol{j}$ are eliminated from these equations it is found that

$$\partial\boldsymbol{B}/\partial t = -\nabla\times(\sigma^{-1}\nabla\times(\mu^{-1}\boldsymbol{B})) + \nabla\times(\boldsymbol{u}\times\boldsymbol{B}). \tag{2.7}$$

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By multiplying this equation scalarly by $\mathrm{d}\boldsymbol{S}$ integrating the resulting expression for $\partial\,|\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}|/\partial t$ over the general closed surface S it may be shown that

$$\mathrm{d}N(S;\,t)/\mathrm{d}t = -2\sum\oint_C[\sigma^{-1}\nabla\times(\mu^{-1}\boldsymbol{B})+(\boldsymbol{v}-\boldsymbol{u})\times\boldsymbol{B}]\cdot\mathrm{d}\boldsymbol{C} \tag{2.8}$$

if $\boldsymbol{v}$ denotes the motion of a general point on S (see Hide 1981*b*). The line integrals are taken over all the one or more closed curves C, vector element of length $\mathrm{d}\boldsymbol{C}$, on S where $\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = 0$, in the sense that keeps the neighbouring region where $\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}$ is positive (negative) on the left (right) when moving in the direction of $\mathrm{d}\boldsymbol{C}$. When S is any material surface S' we have $(\boldsymbol{v}-\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{S} = 0$ everywhere on S'; whence $(\boldsymbol{v}-\boldsymbol{u})\times\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{C} = 0$ and

$$\mathrm{d}N(S';\,t)/\mathrm{d}t = -2\sum\oint_C\sigma^{-1}\nabla\times(\mu^{-1}\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{C} = -2\sum\oint_C\sigma^{-1}\boldsymbol{j}\cdot\mathrm{d}\boldsymbol{C}. \tag{2.9}$$

Dynamo action requires high electrical conductivity (see equation (2.13) below), but equation (2.9) shows that it cannot occur in a perfect conductor, since $\mathrm{d}N(S';\,t)/\mathrm{d}t = 0$ when $\sigma^{-1} = 0$.

As we have already noted, the mathematical difficulties presented by the full magnetohydrodynamic dynamo problem are so severe that most studies to date have been concerned with kinematic dynamo problems of obtaining non-decaying solutions of equation (2.7) when $\boldsymbol{u}$ is specified *a priori*. Such solutions can be found, but only when (*a*)

> *$\boldsymbol{u}$ and $\boldsymbol{B}$ are sufficiently complicated in form (there being only decaying solutions when the configurations of $\boldsymbol{u}$ and $\boldsymbol{B}$ possess a common axis of symmetry)* (2.10)

(cf. (1.1)), and (*b*) the ohmic decay time τ_{d} is so long in comparison with the advective time scale τ_{a}, where

$$\tau_{\mathrm{d}} \equiv L^2\bar{\mu}\bar{\sigma};\quad \tau_{\mathrm{a}} \equiv L/U, \tag{2.11}$$

that the so-called 'magnetic Reynolds number'

$$R \equiv UL\bar{\mu}\bar{\sigma} = \tau_{\mathrm{d}}/\tau_{\mathrm{a}} \tag{2.12}$$

satisfies

$$R > R_*, \tag{2.13}$$

where R_* is typically between 10 and 10^2. (Here U, L, $\bar{\mu}$ and $\bar{\sigma}$ are typical value of the flow speed, length scale, magnetic permeability and electrical conductivity, respectively.)

Kinematic dynamo studies have called attention to the role of the helicity of the motion

$$H \equiv \boldsymbol{u}\cdot\nabla\times\boldsymbol{u} \tag{2.14}$$

in the amplification process. This pseudo-scalar quantity is easy to visualize when it is expressed as the sum of three contributions, each proportional to the rate of change with respect to one of the Cartesian coordinates x_i $(i = 1, 2, 3)$ of the direction made by the projection of $\boldsymbol{u}$ on the local (x_j, x_k) $(j = 2, 3, 1;\ k = 3, 2, 1)$ plane perpendicular to the x_i axis. Thus

$$H = \sum_{i=1}^{3} H_i \quad \text{where} \quad H_i = -(u_j^2+u_k^2)\frac{\partial}{\partial x_i}\tan^{-1}\left(\frac{u_k}{u_j}\right). \tag{2.15}$$

This form of H also has certain physical advantages when dealing with large-scale fluid motions that depart but little from rigid body rotation relative to an inertial frame having angular velocity $\boldsymbol{\Omega}$ about one of the coordinate axes, say x_3, and are therefore dominated by Coriolis

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forces (see § 3). A major contribution to H is then provided by H_3, which satisfies the following equation (Hide 1976):

$$H_3 \approx \frac{1}{2\Omega^2}\{(\boldsymbol{u}\cdot\boldsymbol{g})\,(\boldsymbol{\Omega}\cdot\nabla\theta)-(\boldsymbol{u}\cdot\nabla\theta)\,(\boldsymbol{\Omega}\cdot\boldsymbol{g})\}$$
$$+\frac{\boldsymbol{u}\cdot(\boldsymbol{\Omega}\times \mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t)}{\Omega^2}+\frac{(\boldsymbol{\Omega}\times\boldsymbol{u})}{\bar{\rho}}\cdot(\nabla\times(\nabla\times(\mu^{-1}\boldsymbol{B})\times\boldsymbol{B})) \tag{2.16}$$

(where $\boldsymbol{g}$ denotes the acceleration due to gravity and centripetal effects, $\boldsymbol{u}$ the fluid motion relative to the rotating frame and $\bar{\rho}\theta$ the departure of the nearly uniform density from its mean value $\bar{\rho}$). The first term on the right-hand side of equation (2.16) is zero in the absence of buoyancy forces associated with density variations; the second term is zero when the precessional term vanishes (i.e. when $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t$ is either zero or parallel to $\boldsymbol{\Omega}$); and the third term is zero when there are no Lorentz forces.

3. Basic equations: magnetohydrodynamics

The full dynamo problem requires the simultaneous solutions of the equations of electrodynamics, thermodynamics and hydrodynamics. The equations of electrodynamics give

$$\nabla\cdot\boldsymbol{B} = 0 \tag{3.1}$$

and

$$\partial\boldsymbol{B}/\partial t+(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{B}-(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{u} = (\mu\sigma)^{-1}\,\nabla^2\boldsymbol{B} \tag{3.2}$$

when σ and μ are constant (see equations (2.3) and (2.7)). The equations of thermodynamics (see, for example, Gubbins & Masters 1979) comprise an equation of state relating the density $\rho = \bar{\rho}(1+\theta)$ to the pressure p, temperature and chemical composition, together with equations governing the advection and diffusion of heat and variations in chemical composition. The equations of hydrodynamics express continuity of matter and momentum balance of individual fluid elements. The first of these, $\mathrm{D}\rho/\mathrm{D}t+\rho\nabla\cdot\boldsymbol{u} = 0$ (where $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t+\boldsymbol{u}\cdot\nabla$), reduces to

$$\nabla\cdot\boldsymbol{u} = 0, \tag{3.3}$$

when dynamical effects of fluid compressibility are negligible. The momentum equation

$$\rho\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+2\boldsymbol{\Omega}\times\boldsymbol{u}-\boldsymbol{r}\times\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right) = -\nabla p+\boldsymbol{g}\rho-\nabla\times(\nu\rho\nabla\times\boldsymbol{u})+\nabla\times(\mu^{-1}\boldsymbol{B})\times\boldsymbol{B}$$

reduces to its Boussinesq version

$$2\boldsymbol{\Omega}\times\boldsymbol{u}+\nabla P = \boldsymbol{A}, \tag{3.4a}$$

where

$$\nabla P \equiv \nabla(P/\bar{\rho})-\boldsymbol{g}$$

and

$$\boldsymbol{A} \equiv -\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+\boldsymbol{r}\times\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}+\boldsymbol{g}\theta+\nu\nabla^2\boldsymbol{u}+\frac{\{(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B}-\nabla(B^2/2)\}}{\mu\bar{\rho}} \tag{3.4b}$$

when the kinematic viscosity ν is constant, $\boldsymbol{g}$ greatly exceeds the other acceleration terms and fractional density variations θ are very much less than unity. Taking the curl of equation (3.4a) gives the vorticity equation expressing the local balance of angular momentum of an individual fluid element; thus

$$(2\boldsymbol{\Omega}\cdot\nabla)\,\boldsymbol{u} = -\nabla\times\boldsymbol{A}, \tag{3.5a}$$

where

$$\nabla\times\boldsymbol{A} = -\mathrm{D}\boldsymbol{\xi}/\mathrm{D}t+(\boldsymbol{\xi}\cdot\nabla)\,\boldsymbol{u}-2\,\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t-\boldsymbol{g}\times\nabla\theta+\nu\nabla^2\boldsymbol{\xi}+(\mu\bar{\rho})^{-1}\,\nabla\times(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B} \tag{3.5b}$$

if $\boldsymbol{\xi} \equiv \nabla\times\boldsymbol{u}$.

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Magnetostrophic and geostrophic flows

Order of magnitude estimates of the various terms in equations (3.4) and (3.5) applied to motions in the Earth's liquid core indicate that the relative acceleration term $\mathrm{D}\boldsymbol{u}/\mathrm{D}t \equiv \partial \boldsymbol{u}/\partial t + (\boldsymbol{u}\nabla)\,\boldsymbol{u}$ and the viscous term $\nu\nabla^2\boldsymbol{u}$ are many orders of magnitude less than the Coriolis term $2\boldsymbol{\Omega}\times\boldsymbol{u}$. When the precessional term is also negligible we have the case of *magnetostrophic flow*, with

$$\boldsymbol{A} = \boldsymbol{A}_{\mathrm{m}} \equiv \boldsymbol{g}\theta + (\mu\bar{\rho})^{-1}\{(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B} - \nabla(B^2/2)\} \tag{3.6a}$$

and

$$\nabla\times\boldsymbol{A} = \nabla\times\boldsymbol{A}_{\mathrm{m}} = -\boldsymbol{g}\times\nabla\theta + (\mu\bar{\rho})^{-1}\nabla\times(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B}. \tag{3.6b}$$

When, in addition, the Lorentz term is negligible in comparison with the Coriolis term we have the case of *geostrophic flow*, with

$$\boldsymbol{A} = \boldsymbol{A}_{\mathrm{g}} \equiv \boldsymbol{g}\theta \tag{3.7a}$$

and

$$\nabla\times\boldsymbol{A} = \nabla\times\boldsymbol{A}_{\mathrm{g}} = -\boldsymbol{g}\times\nabla\theta. \tag{3.7b}$$

The vorticity equation (3.5) then yields the 'thermal wind equation'

$$(2\boldsymbol{\Omega}\cdot\nabla)\,\boldsymbol{u} = \boldsymbol{g}\times\nabla\theta \tag{3.8}$$

in the 'baroclinic' case ($\boldsymbol{g}\times\nabla\theta \neq 0$), which reduces to the Proudman–Taylor theorem

$$(2\boldsymbol{\Omega}\cdot\nabla)\,\boldsymbol{u} = 0 \tag{3.9}$$

in the 'barotropic' case ($\boldsymbol{g}\times\nabla\theta = 0$), when surfaces of equal density coincide with geopotential surfaces.

Equations (3.8) and (3.9) are succinct expressions of the powerful gyroscopic constraints on the motion of a fluid of low viscosity that departs but little from solid body rotation with steady angular velocity $\boldsymbol{\Omega}$ when Lorentz forces are negligibly small. Studies of such flows are important in dynamo theory because they provide insight into the initial stages of the amplification process, when Lorentz forces would indeed be small. Equation (3.4) with $\boldsymbol{A} = \boldsymbol{A}_{\mathrm{g}}$ (see equation (3.7)) leads to the important result (cf. (1.2)):

> *The hydrodynamical motion of a fluid of low viscosity that departs only slightly from steady rapid rigid-body rotation will not in general be symmetric about the rotation axis, even when the boundary conditions are axisymmetric.* (3.10)

The validity of this result (which provides the most direct explanation of the occurrence of large-scale non-axisymmetric disturbances in the Earth's atmosphere and other natural systems) is readily verified by laboratory experiments. The result can be deduced as follows. In cylindrical co-ordinates (r, ϕ, z) where $\boldsymbol{\Omega} = (0, 0, \Omega)$ the second component of equation (3.4) is:

$$u_r = (2\Omega)^{-1}\{-r^{-1}\partial P/\partial\phi + (\boldsymbol{A})_{\phi}\} \tag{3.11}$$

(since $(\boldsymbol{g})_{\phi} = 0$ by the assumption of axial symmetry in the boundary conditions), where $(\boldsymbol{A})_{\phi}$ denotes the ϕ component of $\boldsymbol{A}$. Now, over any cylindrical surface of radius r the rate of advective transport $M(r, t;\, Q)$ of any quantity Q (per unit volume), such as heat, angular momentum, etc., is given by

$$M(r, t;\, Q) = \int_{z_1}^{z_2}\int_0^{2\pi} u_r\, Q r\,\mathrm{d}\phi\,\mathrm{d}r = \frac{1}{2\Omega}\int_{z_1}^{z_2}\int_0^{2\pi}\left\{-\frac{\partial P}{\partial\phi} + r(\boldsymbol{A})_{\phi}\right\} Q\,\mathrm{d}\phi\,\mathrm{d}z. \tag{3.12}$$

Since the contribution $(\boldsymbol{A})_{\phi}$ to equation (3.11) decreases rapidly with increasing Ω, advective transport perpendicular to the axis of rotation, as measured by $M(r, t;\, Q)$, will be negligible

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unless the flow pattern departs significantly from axial symmetry. In the axisymmetric case we have $\partial P/\partial \phi = 0$ and $M(r, t; Q)$ of the order of the small ageostrophic contribution.

This argument is the basis of (3.10). There may be singular cases when the flow remains axisymmetric and in consequence advective transfer perpendicular to the rotation axis is negligible. Indeed, such cases can be realized in the laboratory by taking certain special precautions, but the general conclusion from laboratory experiments is that (3.10) is a correct inference from the geostrophic equation.

There is a further property of equation (3.4) with $\boldsymbol{A} = \boldsymbol{A}_g$ that leads to a useful general prediction. The equation is mathematically degenerate; being lower in order than the full equation to which it is a leading approximation when Ω is large, it cannot be solved under all the necessary boundary conditions. For this to be possible every term in $\boldsymbol{A}$ must be included in the analysis, which implies:

Regions of highly ageostrophic flow occurring not only on the boundaries of the system but also in localized regions (detached shear layers, jet streams, etc.) of the main body of the fluid are necessary concomitants of geostrophic motion. (3.13)

Within these highly ageostrophic regions, $\rho\, \mathrm{D}\boldsymbol{u}/\mathrm{D}t + \nabla \times (\rho\nu\nabla \times \boldsymbol{u})$ is comparable in magnitude with $2\rho\boldsymbol{\Omega} \times \boldsymbol{u}$; the corresponding relative vorticity $\boldsymbol{\xi} \equiv \nabla \times \boldsymbol{u}$ can be comparable with or even exceed $2\boldsymbol{\Omega}$ in magnitude. Many examples of such vorticity concentrations are found in the laboratory and in Nature.

We have seen that slow relative hydrodynamical flow in a rotating fluid of low viscosity will in general be non-axisymmetric (see (3.10)). Laboratory studies show that there are two non-axisymmetric régimes of thermal convection in a rotating fluid annulus subject to differential heating in the horizontal, one highly regular (i.e. spatially and temporally periodic) and the other, which is reminiscent of large-scale flow in the Earth's atmosphere, irregular. Thus when the basic rotation rate Ω of the fluid annulus exceeds a certain value Ω_{R}, Coriolis forces inhibit axisymmetric overturning motions in meridian planes and promote a completely different kind of motion, which has been termed 'sloping convection'. The motion is then non-axisymmetric and largely confined to jet-streams, with typical trajectories of individual fluid elements inclined at small but essentially non-zero angles to the horizontal. The kinetic energy of the non-axisymmetric flow derives from the interaction of slight upward and downward motions in these sloping trajectories with the potential energy field produced by the action of gravity on the density variations produced by the applied differential heating. The kinetic energy of the motion is dissipated by friction arising in boundary layers on the walls of the container and in the main body of the fluid. The critical value Ω_{R} of the rotation speed is of course dependent on many parameters, including the acceleration of gravity, the shape and dimensions of the apparatus, the coefficients of thermal expansion, thermal conductivity and viscosity of the fluid and its mean density, and the distribution and intensity of applied differential heating. This dependence has been determined by extensive laboratory studies and interpreted on the basis of stability theory.

Provided that Ω, though greater than Ω_{R}, does not exceed a second critical value Ω_{I}, the main features of the non-axisymmetric motion are characterized by great regularity and the heat flow is virtually independent of Ω and some 20% less than when $\Omega = 0$. This regular flow is either steady (apart from a slow steady drift of the horizontal flow pattern relative to the walls of the container) or it exhibits periodic 'vacillation' in amplitude, shape and other characteristics. The number of 'waves' m around the annulus is not uniquely determined by the impressed

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conditions; the flow is found to be 'intransitive' owing to the occurrence of what are now called 'multiple equilibrium states'. But the most likely value of m tends to increase with increasing Ω, and when $\Omega = \Omega_{\mathrm{I}}$, m has that value for which the azimuthal scale of the horizontal flow pattern is about 1.5 times the radial scale and the flow undergoes a transition to irregular flow or 'geostrophic turbulence'. When $\Omega > \Omega_{\mathrm{I}}$ we have the irregular flow régime, for which heat flow decreases with increasing Ω.

The behaviour just described is now known to be typical of a wide variety of dynamical systems, where large-scale motions can be highly regular under some impressed conditions and highly irregular under other conditions (see, for example, Haken 1981; Hide 1981 *c*, 1982). Both types of large-scale flow can occur in natural systems and are therefore of interest in the theory of magnetic field generation by dynamo action. Underlying mechanisms are not yet fully understood but it is likely that one important role of Coriolis forces is to render the flow highly anisotropic. Energy transfer between different scales of motion within such flows contrasts sharply with that which occurs in isotropic flows, where nonlinear interactions can produce a 'cascade' of energy towards the smallest scales of motion and render the system highly chaotic (i.e. turbulent). Such cascades cannot occur in typical anisotropic flows unless they are accompanied by a simultaneous energy transfer to the largest scales available.

4. Magnetostrophic flows

Setting $Q = 1$ in equation (3.12) leads to a useful general result which, when applied to the magnetostrophic case, reduces to the expression for the constraint on $\boldsymbol{B}$ first noted by Taylor (1963), namely that over any cylindrical surface coaxial with the rotation axis, $\boldsymbol{B}$ must be such that

$$\int_{z_1}^{z_2}\int_0^{2\pi}\{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\}_{\phi}\, r\,\mathrm{d}\phi\,\mathrm{d}z = 0. \tag{4.1}$$

The term in equation (3.12) involving P vanishes when $Q = 1$ because P is single valued. $M(r, t;\, Q)$ is negligibly small when $Q = 1$ because, by considerations of continuity, $M(r, t;\, 1)$ is equal to the volume flux across the surfaces $z = z_1(r)$ and $z = z_2(r)$; this depends on boundary layer suction, which vanishes in the limit of zero viscosity. It follows that

$$\int_{z_1}^{z_2}\int_0^{2\pi}(2\Omega)^{-1}\,(\boldsymbol{A})_{\phi}\, r\,\mathrm{d}\phi\,\mathrm{d}z = 0 \tag{4.2}$$

and this reduces to Taylor's result (see equation (3.1)) in the magnetostrophic case when $\boldsymbol{A} = \boldsymbol{A}_{\mathrm{m}}$ (see equation 3.6*a*)), since $\boldsymbol{g}$ has no ϕ component.

Let us now consider the problem of deducing from first principles the strength of the magnetic field produced by dynamo action, denoting by B_{e} the average field strength just outside the dynamo region and by B_{i} the average strength of the field within the dynamo region. This difficult problem has not yet been solved but it has been discussed by several investigators (for references see Jacobs 1975; Parker 1979). In an attempt to set useful limits on B_{e} and B_{i} Hide (1974) has argued on the basis of general considerations of equations (3.2) and (3.4) that the magnetic field is unlikely to build up beyond that value for which the Lorentz torque $\bar{\rho}\nabla\times(\nabla\times(\mu^{-1}\boldsymbol{B})\times\boldsymbol{B})$ acting on an individual fluid element is unlikely to exceed the acceleration term $\nabla\times(\mathrm{D}\boldsymbol{u}/\mathrm{D}t + 2\boldsymbol{\Omega}\times\boldsymbol{u})$, which reduces to $\nabla\times(2\boldsymbol{\Omega}\times\boldsymbol{u})$ in the magnetostrophic limit. From this, B_{i} satisfies

$$B_{\mathrm{i}} \lesssim B_{\mathrm{s}} R^{\frac{1}{2}}, \tag{4.3}$$

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where R is the magnetic Reynolds number (see equation (2.12)) and B_s is the 'scale magnetic field strength',

$$B_s \equiv \{\bar{\rho}(\Omega + UL^{-1})/\bar{\sigma}\}^{\frac{1}{2}}, \tag{4.4}$$

which reduces to $(\bar{\rho}\Omega/\bar{\sigma})^{\frac{1}{2}}$ in the magnetostrophic limit when $U/L\Omega \ll 1$. The ratio of the magnetic to kinetic energy implied by equation (4.3) is given by

$$B_i^2/\mu U^2 = \Omega L/U + 1. \tag{4.5}$$

This is very much greater than unity when $U/L\Omega \ll 1$, as in the case of a typical planetary dynamo.

Now, although the rate of generation of total magnetic energy by the dynamo mechanism, and hence B_i, is expected to *increase* with increasing electrical conductivity σ, the strength B_e of the *external* magnetic field produced by the dynamo (and this is the only part of the field we are able to observe) should *decrease* with increasing σ when σ is large, with B_e vanishing altogether when σ is infinite, for it is impossible to change the magnetic flux linkage of a perfect conductor (see equation (2.9)). It can therefore be supposed that

$$B_e/B_i \approx R^{-q}, \tag{4.6}$$

where the index q is essentially positive and possibly close to unity. Hence

$$B_e \lesssim B_s R^{(1-2q)/2} \equiv \hat{B}_e. \tag{4.7}$$

$\hat{B}_e$ is thus an upper limit to the strength of the magnetic field just above the fluid region where dynamo action is taking place. If $q \approx 1$, then $\hat{B}_e$ satisfies

$$B_s R^{-\frac{1}{2}} \approx \hat{B}_e \ll B_s R^{\frac{1}{2}}. \tag{4.8}$$

Corresponding expressions for the equivalent magnetic moment can be obtained from equations (4.7) and (4.8) by multiplying by the cube of an appropriate length. Taking as typical values for the core of the Earth

$$\Omega \approx 10^{-4}\,\mathrm{rad\,s^{-1}}, \quad \bar{\rho} \approx 10^4\,\mathrm{kg\,m^{-3}}, \quad \bar{\sigma} \approx 3\times 10^5\,\mathrm{S\,m^{-1}}, \quad U \approx 10^{-4}\,\mathrm{m\,s^{-1}}, \quad \text{and} \quad L \approx 10^6\,\mathrm{m},$$

we find that $U/L\Omega \approx 10^{-6}$ and $B_s \approx (\bar{\rho}\Omega/\bar{\sigma})^{\frac{1}{2}} = 2\times 10^{-3}\,\mathrm{T}\,(20\,\mathrm{G})$. Accordingly, by equations (4.3) and (4.8) we have

$$B_i \lesssim 10^{-2}\,\mathrm{T}\,(100\,\mathrm{G}); \quad B_e \lesssim 4\times 10^{-4}\,\mathrm{T}\,(4\,\mathrm{G})$$

if we assume that, in accordance with kinematic dynamo studies, $R \approx 25$. Now, the mainly dipolar field of $5\times 10^{-5}\,\mathrm{T}\,(0.5\,\mathrm{G})$ at the surface of the Earth implies that the average field strength just outside the core, B_e, is less than about $10^{-3}\,\mathrm{T}\,(10\,\mathrm{G})$. This falls within the preferred range of B_e implied by the above discussion. So far as the value of B_i for the Earth's core is concerned, this cannot be inferred directly from geomagnetic observations. But various lines of evidence indicate that the magnetic field throughout the main body of the core might be largely toroidal in configuration, with the lines of force lying approximately on horizontal surface, and *ca.* $10^{-2}\,\mathrm{T}$ (100 G) in strength (see Hide & Roberts 1979), which is also concordant with the above calculation.

By Faraday's law the magnetic flux linkage of a perfect conductor cannot change so that effects due to ohmic dissipation are central to dynamo theories of the generation of the external magnetic field (see equation (2.9) above). On the other hand, such effects may not be of primary importance when dealing with some aspects of the motions themselves and their neglect leads to a considerable simplification of equation (2.7) (see also equation (3.2)), which then reduces to

$$\partial \boldsymbol{B}/\partial t + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{u} = 0, \tag{4.9}$$

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Alfvén's celebrated 'frozen field' theorem. The mathematical difficulties involved are still severe, especially when realistic boundary conditions are taken into account, and their discussion lies beyond the scope of this article (for references see Soward 1982). Fortunately, some of the main dynamical processes are exemplified by the properties of small-amplitude plane waves with angular frequency ω and vector wavenumber $\boldsymbol{k}$ propagating relative to a fluid that rotates uniformly with steady angular velocity $\boldsymbol{\Omega}$, is pervaded by a uniform magnetic field $\boldsymbol{B}_0$, and within which there is a uniform vertical density gradient $\bar{\rho}\,\mathrm{d}\theta_0/\mathrm{d}z$, where z is the downward vertical coordinate. The dispersion relationship for these waves is

$$\omega^2 = \omega_\mathrm{m}^2 + \tfrac{1}{2}[\omega_\mathrm{s}^2 + \omega_\mathrm{r}^2 \pm \{(\omega_\mathrm{s}^2 + w_\mathrm{r}^2)^2 + 4\omega_\mathrm{m}^2\,\omega_\mathrm{r}^2\}^{\frac{1}{2}}] \tag{4.10}$$

where $$\omega_\mathrm{m}^2 \equiv (\boldsymbol{B}_0\cdot\boldsymbol{k})^2/\mu\bar{\rho}; \quad \omega_\mathrm{s}^2 \equiv -(\boldsymbol{g}\times\boldsymbol{k})^2\,(\mathrm{d}\theta_0/\mathrm{d}z)/k^2; \quad \omega_\mathrm{r}^2 \equiv (2\boldsymbol{\Omega}\cdot\boldsymbol{k})^2/k^2. \tag{4.11}$$

In the three cases when all but one of the quantities ω_m^2, ω_s^2 and ω_r^2 is equal to zero, we have

$$\omega^2 = \omega_\mathrm{m}^2, \quad \omega^2 = \omega_\mathrm{r}^2 \quad \text{or} \quad \omega^2 = \omega_\mathrm{s}^2. \tag{4.12}$$

The first of these expressions is the dispersion relation for ordinary Alfvén waves, where the restoring force is provided entirely by the magnetic field and there is on average equipartition between kinetic and magnetic energy. These waves are non-dispersive and linearly polarized. The second expression is the dispersion relation for inertial waves, where the restoring force is provided by Coriolis effects. These waves are highly dispersive, circularly or elliptically polarized and less than or equal to 2Ω in frequency. The third expression in the dispersion relation for internal gravity waves, where buoyancy forces provide the restoring force when $\mathrm{d}\theta_0/\mathrm{d}z$ is positive (and promote convective overturning when $\mathrm{d}\theta_0/\mathrm{d}z$ is negative). These waves are highly dispersive and linearly polarized and less than or equal to $(-g\,\mathrm{d}\theta_0/\mathrm{d}z)^{\frac{1}{2}}$ in frequency.

In general there are two modes according to whether the upper or lower sign is taken in equation (4.10); we designate these as the 'fast' and 'slow' modes and their frequencies by ω_+ and ω_- respectively, which satisfy

$$\omega_+^2\,\omega_-^2 = \omega_\mathrm{m}^2(\omega_\mathrm{m}^4 + \omega_\mathrm{s}^2) \tag{4.13}$$

for all values of ω_r^2. When $\omega_\mathrm{s}^2 = 0$ we have

$$\omega^2 = \omega_\mathrm{m}^2 + \tfrac{1}{2}[\omega_\mathrm{r}^2 \pm \{\omega_\mathrm{r}^4 + 4\omega_\mathrm{m}^2\,\omega_\mathrm{r}^2\}^{\frac{1}{2}}] \tag{4.14}$$

and $$\omega_+^2\,\omega_-^2 = \omega_\mathrm{m}^4. \tag{4.15}$$

In the case when rotational effects are weak, notably when $\omega_\mathrm{r}^2 \ll 2\omega_\mathrm{m}^2$, equation (4.14) gives

$$\omega_+^2 = \omega_\mathrm{m}^2(1 + |\omega_\mathrm{r}/\omega_\mathrm{m}|); \quad \omega_-^2 = \omega_\mathrm{m}^2(1 - |\omega_\mathrm{r}/\omega_\mathrm{m}|), \tag{4.16}$$

which correspond to ordinary Alfvén waves very slightly modified by Coriolis forces.

At the other extreme, when $\omega_\mathrm{r}^2 \gg 2\omega_\mathrm{m}^2$, and this is the case of most interest when dealing with waves in the core of the Earth on scales of more than a few hundred kilometres, Coriolis forces are so strong that the two roots of equation (4.14) can have quite different values:

$$\omega_+^2 = \omega_\mathrm{r}^2; \quad \omega_-^2 = \omega_\mathrm{m}^4/\omega_\mathrm{r}^2. \tag{4.17}$$

This extreme 'frequency splitting' due to rotation is accompanied by other effects, notably wave dispersion, circular or elliptical polarization of the trajectories of individual fluid elements, and imbalance of kinetic energy (the whole of which is now associated with the fast inertial wave) and magnetic energy (now entirely in the slow 'magnetohydrodynamic inertial' wave).

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When equations (4.17) are satisfied, the period of the inertial mode $2\pi/\omega_+$ is then typically more than about π/Ω (i.e. a few days), whereas that of the magnetic mode $2\pi/\omega_-$ is *ca.* $2\pi\Omega L^2\mu\bar{\rho}/B_0^2$, which for the Earth's core when $L \approx 10^6$ m is 10^{10} s (300 years) and therefore comparable with the timescale of the geomagnetic secular variation. This is the quantitative basis of the theory of the geomagnetic secular variation that interprets its general timescale and westward drift in terms of magnetohydrodynamic oscillations of the liquid core. (The electrical conductivity of the overlying 'solid' mantle, though weak, would be sufficient to prevent magnetic changes in the core on the shortest timescale of the inertial modes from penetrating to the Earth's surface.) These oscillations should play an important role in the dynamo process and also in the electromagnetic and topographic coupling between the liquid core and overlying solid mantle that has been invoked to account for the so-called decade variations in the length of the day (for references see Braginskiy 1980; Hassan & Eltayeb 1982).

The root corresponding to the magnetohydrodynamic inertial wave could have been obtained directly by using the magnetostrophic version of equations (3.4), a procedure that eliminates solutions corresponding to inertial waves from the system of equations *ab initio* (just as the use of equation (3.3) acts as a filter for sound waves in the analysis). When $\boldsymbol{g} \times \nabla\theta = 0$, there is a balance of Coriolis and Lorentz couples acting on individual fluid elements (see equations (3.5) and (3.6)), so that the ratio of amplitudes of the velocity and magnetic fields associated with the wave is $B_0/\Omega L\mu\bar{\rho}$. By equation (4.9) this ratio is also equal to $LB_0\tau_-$, where $\tau_- = 2\pi/\omega_-$, the period of the wave; whence

$$\tau_- \approx \Omega L^2\mu\bar{\rho}/B_0^2, \tag{4.18}$$

which is *ca.* $\{\Omega L/B_0(\mu\rho)^{-\frac{1}{2}}\}^2$ (about 10^5 for the core of the Earth) rotation periods ('days') of the system and *ca.* $\Omega L/B_0(\mu\bar{\rho})^{-\frac{1}{2}}$ multiplied by the time taken for an ordinary Alfvén wave to traverse a distance equal to the characteristic length scale L.

5. Concluding remarks

I have outlined certain general properties of flows that are strongly influenced by Coriolis forces and Lorentz forces which can be deduced in a fairly straightforward way from the basic equations of magnetohydrodynamics. In preparing this brief survey, no attempt has been made to do full justice to the extensive important work that has been done on the magnetohydrodynamics of rotating fluids and dynamo theory, but references to articles describing original work and the development of ideas can be found in the reviews cited in the reference list. My remarks here are addressed primarily to those participants in this Discussion Meeting who are concerned with the structure and evolution of the Earth's core and more practical aspects of the study of geomagnetism, in the hope that the remarks can serve as a guide to the more technical and often highly mathematical literature describing the theory of the generation of cosmical magnetic fields by the self-exciting dynamo process.

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